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In-Plane Vibration Analysis of Curved Beams Considering Shear Deformation Using DQM

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전단변형이론 및 미분구적법을 이용한 곡선보의 내평면 진동해석

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Abstract DQM(differential quadrature method) is applied to computation of eigenvalues of the equations of motion governing the free in-plane vibration for circular curved beams including both rotatory inertia and shear deformation. Fundamental frequencies are calculated for the members with clamped-clamped end conditions and various opening angles. The results are compared with numerical solutions by other methods for cases in which they are available. The differential quadrature method gives good accuracy even when only a limited number of grid points is used.

Key Words: circular curved beam, DQM, fundamental frequencies, rotatory inertia, vibration, shear deformation, numerical solution

요 약 곡선보(curved beam)의 회전관성(rotatory inertia) 및 전단변형(shear deformation)을 고려한 평면내(in-plane) 자유진동을 해석하는데 미분구적법(DQM)을 이용하여 고정-고정 경계조건(boundary conditions)과 다양한 굽힘각(opening angles)에 따른 진동수(frequencies)를 계산하였다. DQM의 결과는 엄밀해(exact solution) 또는 다른 수치해석(Rayleigh-Ritz, Galerkin 또는 FEM) 결과와 비교하였으며, DQM은 적은 요소(grid points)를 사용하여 정확한 해석결과를 보여주었다.

1. Introduction

Curved beams are used frequently in highway bridge structures. Curved alignments of highway bridges and interchanges have been necessary for the smooth dissemination of traffic in large urban areas. The construction cost and time of curved beams associated with the substructure have been found to be significantly reduced by the use of curved beams. Furthermore, the construction time is a factor of immense importance in the selection of a suitable structural system where the construction site needs to be used for other operations during the construction period (Kang and Yoo [1]).

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Owing to their importance in many fields of technology and engineering, the vibration behavior of elastic curved beams has been the subject of a large number of investigations. Despite of a number of advantages, a curved member behaves in an extremely complex manner as compared to a straight member, and practicing engineers have often been discouraged by the complexity because of the initial curvature. However, the mathematical difficulties associated with curved members have been largely overcome with the application of digital computers and the development of numerical methods.

The early investigators into the in-plane vibration of rings were Hoppe [2] and Love [3]. Love [3] improved on Hoppe's theory by allowing for stretching of the ring. Lamb [4] investigated the statics of incomplete ring with various boundary conditions and the dynamics of an incomplete free-free ring of small curvature. Den Hartog [5] used the Rayleigh-Ritz method for finding the lowest natural frequency of circular arcs with clamped ends, and his work was extended by Volterra and Morell [6] for the vibrations of arches having center lines in the form of cycloids, catenaries or parabolas. Archer [7] carried out for a mathematical study of the in-plane inextensional vibrations of an incomplete circular ring of small cross section with the basic equations of motion as given in Love [3] and gave a prescribed time - dependent displacement at the other end for the case of clamped ends. Nelson [8] applied the Rayleigh-Ritz method in conjunction with Lagrangian multipliers to the case of a circular ring segment having simply supported ends. Recently, Irie et al. [9] have analyzed circular arches based on Bresse-Timoshenko beam theory in which both rotatory inertia and shear deformation are taken into account.

A rather efficient alternate procedure for the solution of partial differential equations is the method of differential quadrature which was introduced by Bellman and Casti [10]. This simple direct technique can be applied to a large number of cases to circumvent the difficulties of programming complex algorithms for the computer, as well as excessive use of storage. This method is used in the present work to analyze the free in-plane inextensional and shear deformable vibrations of curved beams with clamped-clamped boundary conditions and various opening angles. The lowest frequencies are calculated for the member. The curved beams considered are of uniform cross section and mass per unit of length. Numerical results are compared with other numerical solutions .

2. System and Governing Equations

The uniform curved beam considered is shown in Figure 1. A point on the centroidal axis is defined by the angle θ, measured from the left support. The tangential and radial displacements of the arch axis are v and w, respectively. a is the radius of the centroidal axis. A mathematical study of the in-plane inextensional vibrations of a curved beam of small cross section is carried out starting with the basic equations of motion as given by Love [3]. Following Love [3], the analysis is simplified by restricting attention to problems where there is no extension of the center line. This condition requires that w and v are related by

$$w = -\frac{\partial v}{\partial \theta} \tag{1}$$

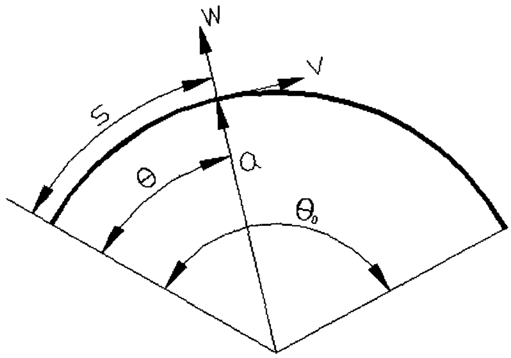

Fig 1. Curved beam considered

If rotatory inertia and shear deformation are neglected, the differential equation governing the free flexural vibration of this curved beam, in terms of the displacement v, can be written as

$$\frac{EI_x}{a^4}\left(\frac{\partial^6 v}{\partial \theta^6}+2\frac{\partial^4 v}{\partial \theta^4}+\frac{\partial^2 v}{\partial \theta^2}\right)=m\frac{\partial^2}{\partial t^2}\left(v-\frac{\partial^2 v}{\partial \theta^2}\right) \tag{2}$$

or

$$\frac{v^{vi}}{\theta_0^6}+2\frac{v^{iv}}{\theta_0^4}+\frac{v''}{\theta_0^2}=\frac{ma^4\omega^2}{EI_x}\left(\frac{v''}{\theta_0^2}-v\right) \tag{3}$$

in which each prime denotes one differentiation with respect to the dimensionless distance coordinate, X, defined as

$$X=\frac{\theta}{\theta_0} \tag{4}$$

Here, m is the mass per unit length, θ_0 is the opening angle for the curved beam, ω is the circular frequency of vibration of the system, E is the Young's modulus of elasticity for the material of the arch and I_x is the area moment of inertia of the cross section.

If the curved beam is clamped at $\theta=0$ and $\theta=\theta_0$, then the boundary conditions take the form

$$v=0 \tag{5}$$

$$w=-\frac{\partial v}{\partial \theta}=0 \tag{6}$$

$$\frac{\partial^2 v}{\partial \theta^2}+v=0 \tag{7}$$

at $\theta=0$ and $\theta=\theta_0$ or

$$v(0)=v'(0)=v''(0)=v(\theta_0)=v'(\theta_0)=v''(\theta_0)=0 \tag{8}$$

The differential equations governing the in-plane vibration of a circular arch based on the Bresse-Timoshenko beam theory, in which both rotatory inertia and shear deformation are taken into account were given by Irie et al. [9] as

$$\frac{k}{2(1+\nu)}\frac{w''}{a\theta_0^2}-(1-\lambda^2\frac{1+k^2}{s_x^2})\frac{w}{a}-\frac{k}{2(1+\nu)}\frac{\psi'}{\theta_0}+(1+\frac{k}{2(1+\nu)})\frac{v'}{a\theta_0}=0 \tag{9}$$

$$\frac{k}{2(1+\nu)}\frac{w'}{a\theta_0}+k^2\frac{\psi''}{\theta_0^2}-(\frac{k}{2(1+\nu)}-\lambda^2\frac{k_2^2}{s_x^2})\psi+(\frac{k}{2(1+\nu)}+\lambda^2\frac{k_1^2}{s_x^2})\frac{v}{a}=0 \tag{10}$$

$$(1+\frac{k}{2(1+\nu)})\frac{w'}{a\theta_0}-(\frac{k}{2(1+\nu)}+\lambda^2\frac{k_1^2}{s_x^2})\psi-\frac{v''}{a\theta_0^2}+(\frac{k}{2(1+\nu)}-\lambda^2\frac{1+k^2}{s_x^2})\frac{v}{a}=0 \tag{11}$$

Here, k is the shear correction factor depending on the shape of the cross section, ν is the Poisson's ratio of the arch and ψ is the slope of the displacement curve due to pure bending. For simplicity of the analysis, the following dimensionless variables have been introduced:

$$s_x^2=A\,a^2/I_x, \quad \lambda^2=ma^4\omega^2/EI_x \tag{12}$$

where s_x is the slenderness ratio of the arch. The quantities k^2, k_1^2 and k_2^2 are the dimensionless parameters defined as

$$k^2=(d/4a)^2, \quad k_1^2=k^2(1+k^2), \quad k_2^2=k^2(1+4k^2+k^4) \tag{13}$$

for an arch with circular cross section of diameter d and

$$k^2=(h/2a)\coth(h/2a)-1 \tag{14}$$

$$k_1^2=k^2(1+k^2)+(1/3)(h/2a)^2 \tag{15}$$

$$k_2^2=k^2[k^2+k^4+(h/2a)^2]+(1/3)(h/2a)^2 \tag{16}$$

for an arch with rectangular cross section of height h. If the arch is clamped at $\theta=0$ and $\theta=\theta_0$, then the boundary conditions take the form

$$w(0)=\psi(0)=v(0)=w(\theta_0)=\psi(\theta_0)=v(\theta_0)=0 \tag{17}$$

3. Differential Quadrature Method

The differential quadrature method (DQM) was introduced by Bellman and Casti [10]. By formulating the quadrature rule for a derivative as an analogous extension of quadrature for integrals in their introductory paper, they proposed the differential quadrature method as a new technique for the numerical solution of initial value problems of ordinary and partial differential equations. It was applied for the first time to static analysis of structural components by Jang et al. [11]. The versatility of the DQM to engineering analysis in general and to structural analysis in particular is becoming increasingly evident by the related publications of recent years. Kang and Han [12] applied the method to classical and shear deformable theories of circular curved beams. Kang [13] and Kang and Kim [14] studied the vibration analysis of curved beams using DQM. From a mathematical point of view, the application of the differential quadrature method to a partial differential equation can be expressed as follows:

$$L\{f(x)\}_i = \sum_{j=1}^{N} W_{ij} f(x_j) \quad \text{for } i, j = 1, 2, 3, \ldots, N \tag{18}$$

where L denotes a differential operator, x_j is the discrete points considered in the domain, $f(x_j)$ is the function values at these points, W_{ij} is the weighting coefficients attached to these function values and N denotes the number of discrete points in the domain. This equation, thus, can be expressed as the derivatives of a function at a discrete point in terms of the function values at all discrete points in the variable domain.

The general form of the function $f(x)$ is taken as

$$f_k(x) = x^{k-1} \quad \text{for} \quad k = 1, 2, 3, \ldots, N \tag{19}$$

If the differential operator L represents an n^{th} derivative, then

$$\sum_{j=1}^{N} W_{ij} x_j^{k-1} = (k-1)(k-2)\cdots(k-n)x_i^{k-n-1}$$
$$\text{for} \quad i, k = 1, 2, 3, \ldots, N \tag{20}$$

This expression represents N sets of N linear algebraic equations, giving a unique solution for the weighting coefficients, W_{ij}, since the coefficient matrix is a Vandermonde matrix which always has an inverse, as described by Hamming [15].

4. Application

Applying the differential quadrature method to equation (3) gives

$$\frac{1}{\theta_0^6}\sum_{j=1}^{N} F_{ij}v_j + \frac{2}{\theta_0^4}\sum_{j=1}^{N} D_{ij}v_j + \frac{1}{\theta_0^2}\sum_{j=1}^{N} B_{ij}v_j = \frac{ma^4\omega^2}{EI_x}\left(\frac{1}{\theta_0^2}\sum_{j=1}^{N} B_{ij}v_j - v_i\right) \tag{21}$$

where B_{ij}, D_{ij} and F_{ij} are the weighting coefficients for the second-, fourth- and sixth-order derivatives, respectively, along the dimensionless axis.

The boundary conditions for clamped ends, given by equation (8), can be expressed in differential quadrature form as follows:

$$v_1 = 0 \quad \text{at} \quad X = 0 \tag{22}$$

$$v_N = 0 \quad \text{at} \quad X = 1 \tag{23}$$

$$\sum_{j=1}^{N} A_{2j}v_j = 0 \quad \text{at} \quad X = 0 + \delta \tag{24}$$

$$\sum_{j=1}^{N} A_{(N-1)j}v_j = 0 \quad \text{at} \quad X = 1 - \delta \tag{25}$$

$$\sum_{j=1}^{N} B_{3j}v_j = 0 \quad \text{at} \quad X = 0 + 2\delta \tag{26}$$

$$\sum_{j=1}^{N} B_{(N-2)j}v_j = 0 \quad \text{at} \quad X = 1 - 2\delta \tag{27}$$

Applying the differential quadrature method to equations (9), (10) and (11) gives

$$\frac{k}{2(1+\nu)}\frac{1}{a\theta_0^2}\sum_{j=1}^{N} B_{ij}w_j - (1 - \lambda^2\frac{1+k^2}{s_x^2}\frac{1}{a}w_i - \frac{k}{2(1+\nu)}\frac{1}{\theta_0}\sum_{j=1}^{N} A_{ij}\psi_j$$
$$+ [1 + \frac{k}{2(1+v)}]\frac{1}{a\theta_0}\sum_{j=1}^{N} A_{ij}v_j = 0 \tag{28}$$

$$\frac{k}{2(1+\nu)}\frac{1}{a\theta_0}\sum_{j=1}^{N} A_{ij}w_j + k^2\frac{1}{\theta_0^2}\sum_{j=1}^{N} B_{ij}\psi_j - [\frac{k}{2(1+\nu)} - \lambda^2\frac{k_2^2}{s_x^2}]\psi_i$$
$$+ [\frac{k}{2(1+\nu)} + \lambda^2\frac{k_1^2}{s_y^2}]\frac{1}{a}v_i = 0 \tag{29}$$

$$[1 + \frac{k}{2(1+\nu)}]\frac{1}{a\theta_0}\sum_{j=1}^{N} A_{ij}w_j - [\frac{k}{2(1+\nu)} + \lambda^2\frac{k_1^2}{s_y^2}]\psi_i - \frac{1}{a\theta_0^2}\sum_{j=1}^{N} B_{ij}v_j$$
$$+ [\frac{k}{2(1+\nu)} - \lambda^2\frac{1+k^2}{s_x^2}]\frac{1}{a}v_i = 0 \tag{30}$$

The boundary conditions for clamped ends, given by equations (17), can be expressed in differential quadrature form as follows:

$$w_1 = \Psi_1 = v_1 = 0 \text{ at } X = 0 \tag{31}$$

$$w_N = \Psi_N = v_N = 0 \text{ at } X = 1 \tag{32}$$

This set of equations together with the appropriate boundary conditions can be solved to obtain the fundamental natural frequency for in-plane vibration of a circular arch.

5. Numerical Results and Comparisons

The fundamental frequency parameter λ of this curved beam is calculated by differential quadrature and is presented together with results from other methods including shear deformation.

Tables 1 and 2 present the results of convergence studies relative to the number of grid points N and the δ parameter, respectively. Table 1 shows that the accuracy of the numerical solution increases with increasing N and passes through a maximum. Then, numerical instabilities arise if N becomes too large. The optimal value for N is found to be 11 to 13. Table 2 shows the sensitivity of the numerical solution to the choice of δ. The optimal value for δ is found to be 1×10^{-5} to 1×10^{-6}, which is obtained from trial-and-error calculations. The solution accuracy decreases due to numerical instabilities if δ becomes too small. All results are calculated using 13 grid points and $\delta = 1\times 10^{-5}$.

Auciello and De Rosa [16] determined the natural frequencies of the arches using the SAP IV or SAP 90 finite element method (FEM). Exact solutions were carried out by Archer [7]. Table 3 shows that the numerical results by the DQM are in excellent agreement with those by the SAP IV FEM and those by exact solutions in the case of neglecting rotatory inertia and shear deformation. However, the SAP IV FEM was quite expensive because 60 finite elements were employed, as described by Auciello and De Rosa [16]. The results are summarized in Table 3.

The values λ corresponding to the lowest natural frequencies are evaluated for circular arches of rectangular and circular cross-sections under clamped-clamped end conditions including shear deformation and numerical results are compared with transfer matrix solutions by Irie et al. [9]. The shear correction factor, k is taken to be 0.85 for the rectangular cross section and 0.89 for the circular cross-section and the Poisson's ratio of the arch, ν, is 0.3. The results are summarized in Tables 4 and 5. As it can be seen, the numerical results show excellent agreement with the solutions by Irie et al. [9] except 12.57* in Table 5. According to Irie et al. [9], the frequency parameters of rectangular cross-section arches are generally smaller than those of circular cross-section arches and the difference between them is very small. It seems, therefore, that 12.57* should be 10.57. From Table 6, the higher values of s_x and θ_0 have little effect on the fundamental natural frequency parameters for both cases of neglecting shear deformation and including shear deformation. However, the lower values of s_x and θ_0 have a significant effect on the frequencies for both cases. In general, as the slenderness ratios of a beam cross section become smaller, the frequencies become more significant.

[Table 1] Fundamental frequency parameters, $\lambda = (ma^4w^2/EI_x)^{1/2}$, for in-plane vibration of thin curved beams with clamped ends including a range of grid point, $\theta_0 = 180^0$

Archer [7] (Exact)	Number of grid points			
$\lambda = (ma^4w^2/EI_x)^{1/2}$	7	9	11	13
4.3841	5.0586	4.1740	4.3975	4.3844

[Table 2] Fundamental frequency parameters, $\lambda = (ma^4w^2/EI_x)^{1/2}$, for in-plane vibration of thin curved beams with clamped ends including a range of δ, $\theta_0 = 180^0$

Archer [7] (Exact)	δ				
$\lambda = (ma^4w^2/EI_x)^{1/2}$	1×10^{-2}	1×10^{-3}	1×10^{-4}	1×10^{-5}	1×10^{-6}
4.3841	4.8845	4.4301	4.3885	4.3844	4.3840

[Table 3] Fundamental frequency parameters, $\lambda = (ma^4w^2/EI_x)^{1/2}$, for in-plane vibration of thin curved beams with clamped ends

θ_0, degrees	$\lambda = (ma^4 w^2/EI_x)^{1/2}$				
	Archer [7] (Exact)	Galerkin	Rayleigh-Ritz	SAP IV finite element	DQM
30		228.18	222.36	222.36	222.36
60		55.221			53.737
90		23.295			22.624
120		12.225			11.847
150		7.194			6.958
180	4.384	4.539			4.384
270	1.395				1.395
324	0.789				0.789
360	0.566				0.566

[Table 4] Fundamental frequency parameter of in-plane vibration $\lambda = (ma^4w^2/EI_x)^{1/2}$ for clamped-clamped arches with circular cross-section including shear deformation; $\nu = 0.3$

S_x	Θ_0(degrees)	Irie et al. [9]	DQM
20	60	23.75	23.758
	120	10.61	10.613
	180	4.151	4.1543
100	60	52.82	52.827
	120	11.79	11.793
	180	4.375	4.3757

[Table 5] Fundamental frequency parameter of in-plane vibration $\lambda = (ma^4w^2/EI_x)^{1/2}$ for clamped-clamped arches with rectangular cross-section including shear deformation; $\nu = 0.3$

S_x	Θ_0(degrees)	Irie et al. [9]	DQM
20	60	23.70	23.709
	120	12.57*	10.585
	180	4.143	4.1478
100	60	52.78	52.795
	120	11.79	11.792
	180	4.374	4.3755

[Table 6] Fundamental frequency parameter of in-plane vibration $\lambda = (ma^4w^2/EI_x)^{1/2}$ for clamped-clamped arches with circular cross-section neglecting shear deformation and including shear deformation using DQM; ν = 0.3

θ_0, degrees	Neglecting shear deformation	Including shear deformation	
		s_x = 20	s_x = 100
60	53.74	23.76	52.83
120	11.85	10.61	11.79
180	4.384	4.154	4.376

6. Conclusions

The differential quadrature method was used to compute the eigenvalues of the equations of motion governing the free in-plane inextensional and shear deformable vibrations of curved beams. The present method gives results which agree very well with the numerical solutions by other methods for the cases treated while requiring only a limited number of grid points.

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