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A Construction of the Principal Period-2 Component in the Degree-9 Bifurcation Set with Parametric Boundaries

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9차 분기집합의 2-주기 성분의 경계방정식에 관한 연구

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Abstract By extending the Mandelbrot set for the complex polynomial

$$M=\left\{c\in C\ :\ \lim_{k\to\infty}P_c{}^k(0)\ \neq\ \infty\right\}$$

we define the degree-n bifurcation set. In this paper, we formulate the boundary equation of a period-2 component on the main component in the degree-9 bifurcation set by parameterizing its image. We establish an algorithm constructing a period-2 component in the degree-9 bifurcation set and the typical implementations show the satisfactory result with Mathematica codes grounded on the analysis.

Key words : degree-n bifurcation set, principal period-2 component, ray of symmetry

요 약 본 논문은 맨델브로트 집합을 9차 복소 다항식에 확장시켜 새로운 프랙탈 도형을 나타내는 9차 분기집합을 정의하고, 2주기 성분의 경계방정식을 매개함수로 표현한다. 또한, 2주기 성분을 작도하는 알고리즘을 고안하고, 매스매티카를 활용하여 2주기 성분의 기하학적 구조에 관한 결과를 제시하고자 한다.

1. Introduction

The Mandelbrot set[1,2] is defined to be the set of all complex values of c such that the critical orbit under the complex polynomial $P_c(z)=z^2+c$ does not escape to infinity, where $c\in C$ denotes the sets of complex numbers. The notion of the Mandelbrot set was generalized to a complex polynomial of degree $n\geqq 2$ and this generalized Mandelbrot set is called the degree-n bifurcation set introduced by Devaney[1]. We investigate the boundary equation of M and the boundary of the principal period-2 bulb in this proposed set.

As beginning studies, we will describe some preliminary definitions and theorems concerning the geometric properties of the degree-n bifurcation set. Let $\boldsymbol{R}$ and $\boldsymbol{N}$ denote the sets of real numbers and natural numbers, respectively.

Definition 1.1 Let $P_c(z)=z^n+c$ for an integer $n\geqq 2$ with $c, z\in C$. Then the degree-n bifurcation set is defined to be the set

$$M=\left\{c\in C\!:\ \lim_{k\to\infty}P_c{}^k(0)\ \neq\infty\right\}$$

where $P^{k+1}(z)=P(P^k(z))$ is the k-fold composite map of P at z with $P^0(z)=z$. This definition was introduced by Devaney[1] in 1986.

Definition 1.2 The sets

$P_m=\left\{c\in C\!:\ c=re^{i\phi_m}, r\geqq 0, \phi_m=m\pi/(n-1)\right\}$ for $(m=1,2,\cdots,2n-2)$ are called the rays of symmetry and P_1 is called the principal ray of symmetry and

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denoted by P. The set

$S=\{c\in C: c=re^{i\theta}, r\geqq 0, 0<\theta\leqq \pi/(n-1)\}$

is called the principal sector.

Definition 1.3 An attracting period-k component[3] is defined by the following set

$$\left\{c\in C: \text{there exist } \xi \text{ such that } P^k_{c}(\xi)=\xi, \left|\frac{d}{dz}P_c^k(z)\right|_{z=\xi}<1\right\}$$

and is denoted by M_k'. Because the attracting period-2 component M_2' lies on every odd ray of symmetry, the M_2' lying on the principal ray of symmetry is called the principal period-2 component

The following theorems convinces the symmetry of the degree-n bifurcation set with respect to rays of symmetry in the complex plane.

Theorems 1.1 The degree-n bifurcation set M is symmetric about rays P_m for all$(m=1,2,\cdots,2n-2)$ in the c-parameter plane.

Proof. See p. 224, Geum and Kim[4,5].

2. Formulating the Boundary of a Period-2 Component

In this section, we will describe the boundary equation of a period-2 component on the the main component in the degree-9 bifurcation set by parameterization. Let $P_c(w)=w^n+c$ with $z, c, w\in C$. The boundary equation of this propose component is as follows:

$$P_c^2(z)=z \qquad (1)$$

$$|\lambda|=1 \qquad (2)$$

where $\lambda=\left|\frac{d}{dw}P_c^2(w)\right|_{w=z}$ is a multiplier at z, which is a period-2 point of P_c. Since (1) describes a circle in the image plane, we use a parameter $\phi\in[0,2\pi)$. From the complex analysis, (2) is represented by

$$z(z^9+c)=\left(\frac{1}{9^2}\right)^{1/8}e^{i\psi_j}, \quad \text{for } j\in 0,1,2,\cdots 7. \qquad (3)$$

where $\psi_j=\frac{\phi+2j\pi}{8}$. Let $a=(1/9^2)^{\frac{1}{8}}e^{i\psi_j}$, for $j\in\{0,1,2,\cdots,7\}$. Then we obtain the following equation for

$$z(z^9+c)=a \qquad (4)$$

Combining (1) with (4), we get $c=z-(a/z)^9=a/z-z^9$ and $z^9-(a/z)^9+z-a/z=0$.

$$\left(z-\frac{a}{z}\right)\left(z^8+z^7\left(\frac{a}{z}\right)+z^6\left(\frac{a}{z}\right)^2+\cdots+\left(\frac{a}{z}\right)^8+1\right)=0$$

Since $z-a/z=0$ means a fixed point, we choose the period-2 point of P_c satisfying the following:

$$z^8+z^7\left(\frac{a}{z}\right)+z^6\left(\frac{a}{z}\right)^2+\cdots+\left(\frac{a}{z}\right)^8+1=0 \qquad (5)$$

Regrouping (5) yields

$$z^8+\frac{a^8}{z^8}+a\left(z^6+\frac{a^6}{z^6}\right)+a^2\left(z^4+\frac{a^4}{z^4}\right)+\cdots+a^3\left(z^2+\frac{a^2}{z^2}\right)+a^4+1=0.$$

From the point c on the boundary satisfies $c=a/z-z^9=a/z-z(z^8)$, we have as follows:

$$c=a\left(z^7+\frac{a^7}{z^7}\right)+a^2\left(z^5+\frac{a^5}{z^5}\right)+\cdots+a^4\left(z+\frac{a}{z}\right)+\left(z+\frac{a}{z}\right) \qquad (6)$$

By Vieta's transformation $t=z+a/z$, we express Eq.(6) simply. We define the function F_k for $k\geqq 0$ by means of the equation

$$F_k=z^k+\frac{a^k}{z^k},\ F_0=2, F_1=z+\frac{a}{z} \qquad (7)$$

It is evident from Eq(7) that

$$\left(z^{k-1}+\frac{a^{k-1}}{z^{k-1}}\right)\left(z+\frac{a}{z}\right)=z^k+\frac{a^k}{z^k}+a\left(z^{k-2}+\frac{a^{k-2}}{z^{k-2}}\right) \qquad (8)$$

That is,

$$F_k=t\cdot F_{k-1}-a\cdot F_{k-2}, \quad \text{for} \quad k\geqq 2. \qquad (9)$$

Evidently, we obtain for $k\geqq 2$,

$$F_k(t,a)=t^k-a\sum_{j=0}^{k-2}t^{k-2-j}\cdot F_j. \qquad (10)$$

We shall derive the following proposition by the induction $m\geqq 1$.

Proposition 1.1 Let $s=t^2-2a$ for fixed a and $s, t, a\in C$. Let $G_m(s)$ and $H_m(s)$ be polynomials in s of degree m and defined by means of the equations

$$\begin{cases} G_{m+1}(s) & = (s+2a)H_m(s) - aG_m(s) \\ & H_{m+1}(s) \\ = G_{m+1}(s) - aH_m(s) \end{cases}$$

where

$G_1 = F_2 = s,\ H_0 = 1,$ and $H_1 = G_1 - aH_0 = s - a.$

Then from definition F_m, it follows that, for $m \geqq 1$

$$\begin{cases} F_{2m} & = G_m(s), \\ F_{2m+1} & = t \cdot H_m(s) \end{cases} \tag{11}$$

The above properties leads us to simplify the system of equations.

According to the equation(11), we can write Eq(6) as

$$c = t\left(1 + \sum_{\nu=1}^{4} a^{\nu} \cdot H_{4-\nu}(s)\right) \tag{12}$$

We find that $c(t_i(a))$ and $c(s_i(a))$ as a function of $a(\psi)$ is each branch of c in the statement of Eq.(12). The symmetry of the degree-9 bifurcation set $\boldsymbol{M}$ with respect to the rays $\frac{m\pi}{8}$, for $m = 1,\ 2, \cdots, 16$ and the continuity of the boundary enable us to consider 16 circular arcs lying on the ray $\frac{m\pi}{8}$.

Let $n \in \boldsymbol{N} - \{1\}$. Now the circular arcs are defined by writing

$$\Gamma_j = \left\{ a \in \boldsymbol{C}\ : a = \left(\frac{1}{9^2}\right)^{1/8} e^{i\theta},\ \frac{j\pi}{4} \leq \theta < \frac{(j+1)\pi}{4} \right\} \quad \text{for } j = 0, 1, \cdots, 7.$$

and $c(t_i(a))$ has 4 circular arcs.

We are interested in the period-2 component lying on the principal ray of symmetry. It is enough to choose $a \in \Gamma_0$ to consider the half of the boundary of the period-2 component.

3. Algorithm and Results

We establish an algorithm drawing the boundaries of period-2 components in this proposed set with Mathematica[6].

Algorithm 3.1

Step 1. Find the respective solution $t_i(a),\ s_i(a)$ of Eq. (5).

Step 2. Evaluate each branch c_i, for $i = 1, 2, \cdots, 4$ of c based on t_i, s_i found in Step 1.

Step 3. Put $x_i = Re(c_i(a(\psi))$ and $y_i = Im(c_i(a(\psi))$ for $i = 1, 2, \cdots, 4$. Plot the curves c_i as the set

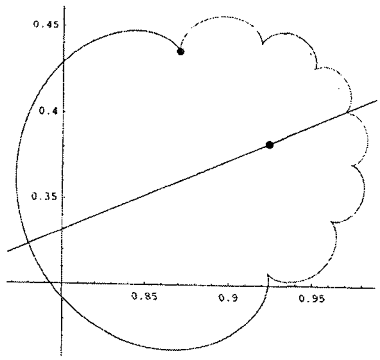
$$\left\{ (x_i(\psi), y_i(\psi)) :\ 0 \leqq \psi \leqq \frac{\pi}{4},\ 1 \leq i \leq 4 \right\}$$

Step 4. Rotate c_i obtained in Step 3 by an angle $\frac{j\pi}{4}$ for $j \in \{0,1,2,3\}$. The rotated curves denoted by G_1 is a part of the boundary of the principal period-2 component.

Step 5. Construct G_2, the symmetric curves of G_1.

Step 6. Complete the boundary curve $G_1 \cup G_2$.

The statements in Section 2 are used to draw the principal period-2 component. Typical boundary and branches of M_2' are shown in Figure 1 with $n = 9$.



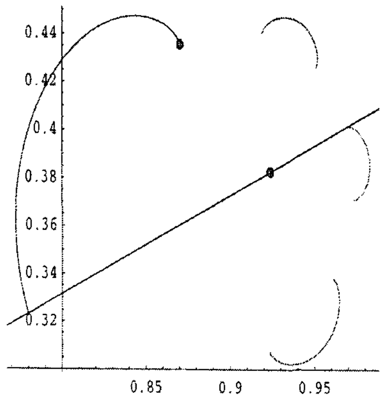


Fig. 1 Typical boundary and branches of M_2'

The current analysis shown in this paper can be departmentalized to parameterize the boundary equation of the period-3 component in the degree-9 bifurcation.

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<Major Research Area>

Fractal sets, Numerical Analysis, Degree-n Bifurcation set, root-finding algorithm,.....