

In-Plane Buckling Analysis of Curved Beams Using DQM

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미분구적법(DQM)을 이용한 곡선보의 내평면 좌굴해석

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Abstract The differential quadrature method (DQM) is applied to computation of the eigenvalues of in-plane buckling of the curved beams. Critical moments and loads are calculated for the beam subjected to equal and opposite bending moments and uniformly distributed radial loads with various end conditions and opening angles. Results are compared with existing exact solutions where available. The DQM gives good accuracy even when only a limited number of grid points is used. More results are given for two sets of boundary conditions not considered by previous investigators for in-plane buckling: clamped-clamped and simply supported-clamped ends.

요 약 곡선보 (curved beam)의 내평면 모멘트 및 등분포하중 하에서 평면내 (in-plane) 좌굴 (buckling)을 미분구적법(DQM)을 이용하여 해석하였다. 다양한 경계조건 (boundary conditions)과 굽힘각 (opening angles)에 따른 임계모멘트 및 임계하중을 계산하였다. DQM의 해석결과는 해석적 해답 (exact solution) 결과와 비교하였으며, DQM은 적은 요소 (grid points)를 사용하여 정확한 해석결과를 보여주었다. 두 경계조건(고정-고정, 단순지지-고정)하에서 새 결과를 또한 제시하였다.

Key Words : Critical Load, Critical Moment, Curved Beam, DQM, Exact Solution, In-Plane Buckling

1. Introduction

Owing to their importance in many fields of technology and engineering, the stability behavior of elastic curved beams has been the subject of a large number of investigations. Solutions of the relevant differential equations have traditionally been obtained by the standard finite difference or finite element methods. These techniques require a great deal of computer time as the number of discrete nodes becomes relatively large under conditions of complex geometry and loading. In a large number of cases, the moderately accurate solution which can be calculated rapidly is desired at only a few

points in the physical domain. However, in order to get results with even only limited accuracy at or near a point of interest for a reasonably complicated problem, solutions often have dependence of the accuracy and stability of the mentioned methods on the nature and refinement of the discretization of the domain.

Ojalvo et al. [1] studied the elastic stability of ring segments with a thrust or a pull directed along the chord. *Vlasov* [2] derived closed-form solutions such as for an arch, in which cross-sections are allowed to warp non-uniformly along the beam axis, subject to in-plane bending and uniformly distributed radial loads. *Papangelis and Trahair* [3] conducted a theoretical study of the

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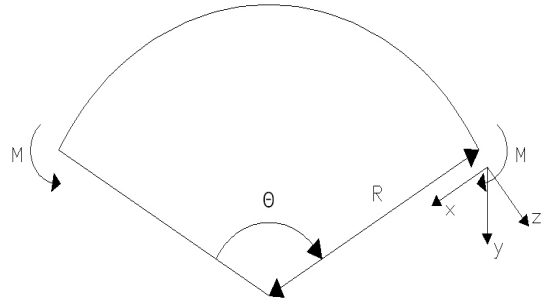
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flexural-torsional buckling of doubly symmetric arches to confirm the predictions of *Timoshenko and Gere* [4] for arches in uniform compression and of *Vlasov* [2] for arches in uniform bending. *Yang and Kuo* [5] studied the static stability of curved beams using the principle of virtual displacements in a Lagrangian formulation with emphasis place on the effect of curvature, and they presented closed-form solutions for arches in uniform bending and uniform compression. In addition, different approaches were also presented by *Kuo and Yang* [6] to support their studies treating a curved beam as the composition of an infinite number of infinitesimal straight beams. *Kang and Yoo* [7] presented a theoretical study on the buckling of curved beams with the derivation of stability equations. Recently, *Kang and Kim* [8] studied the in-plane vibration analysis of asymmetric curved beams using DQM.

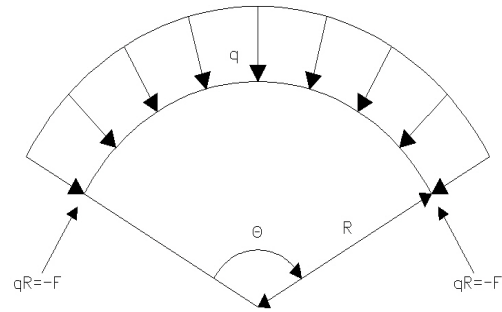
In the present work, the DQM which is a rather efficient alternate procedure for the solution of partial differential equations, introduced by *Bellman and Casti* [9], is use to analyze in-plane buckling of the curved beams subjected to uniformly distributed radial loads and equal and opposite in-plane bending moments. The critical loads and moments are calculated for the member. The curved beams considered are of uniform cross section, and have both ends either simply supported or clamped, or have simply supported-clamped ends. Numerical results are compared with existing exact solutions where available.

2. Governing Differential Equations

The uniform curved beam considered is shown in Figs. 1 and 2 under the equal and opposite bending moment M and the uniform in ward radial pressure q . A point on the centroidal axis is defined by the angle Θ , measured from the left support, and the radius of the centroidal axis is R . The tangential and radial displacements of the arch axis are w and u in the direction of z and x , respectively.



[Fig. 1] Coordinate system of curved beam in-plane bending moment



[Fig. 2] Curved beam with uniformly distributed radial loads

A mathematical study of the in-plane inextensional condition of small cross section is carried out starting with the basic equations as given by *Love* [10]. Following *Love*, the analysis is simplified by restricting attention to problems where there is no extension of the center line. This condition requires that u and w be related by

$$w' - \frac{1}{R}u = 0 \quad (1)$$

If the external forces are assumed to rotate with the centroidal axis of the arch during the process of buckling, and shear deformation is neglected, the differential equation of curved beam subjected to uniformly distributed radial loads can be written as (*Kang and Yoo* [7])

$$\begin{aligned} & -\frac{1}{R}EA\left(w' - \frac{1}{R}u\right) + EI\left(u^{IV} + \frac{2}{R^2}u'' + \frac{1}{R^3}w'\right) \\ & - qR\left(u'' + \frac{1}{R}w' - \frac{1}{R^2}u\right) = 0 \end{aligned} \quad (2)$$

where E is the Young's modulus of elasticity for the material of the arch, and I is the area moment of inertia of the cross section.

Using equation (1) and the dimensionless distance coordinate $X = \theta/\theta_0$, in which θ_0 is the opening angle of the member, one can rewrite equation (2) as

$$\frac{w^V}{\theta_0^5} + 2\frac{w'''}{\theta_0^3} + \frac{w'}{\theta_0} = \left(\frac{qR^3}{EI}\right)\frac{w'''}{\theta_0^3} \quad (3)$$

where each prime denotes one differentiation with respect to the dimensionless distance coordinate.

The differential equation of curved beam subjected to equal and opposite in-plane bending moments can be written as (Kang and Yoo [7]).

$$\begin{aligned} -\frac{1}{R}EA\left(w' - \frac{1}{R}u\right) + EI\left(u^{IV} - \frac{2}{R^2}u'' + \frac{1}{R^3}w'\right) \\ - \frac{M}{R}\left(u'' + \frac{1}{R}w' - \frac{1}{R^2}u\right) = 0 \end{aligned} \quad (4)$$

where M is applied in-plane bending moments.

Using equation (1) and the dimensionless distance coordinate $X = \Theta/\Theta_0$, one can also rewrite equation (4) as

$$\frac{w^V}{\theta_0^5} + 2\frac{w'''}{\theta_0^3} + \frac{w'}{\theta_0} = \left(\frac{MR}{EI}\right)\left(\frac{w'''}{\theta_0^3} + \frac{2}{\theta}w'\right) \quad (5)$$

Since the same inextensibility condition was employed for the various studies, the differences seem to arise only from different formulations of the stability equations by other investigators. The differential equations of curved beam subjected to equal and opposite in-plane bending moments and subjected to uniformly distributed radial loads by Yang and Kuo [5] can be written, respectively.

$$\frac{w^V}{\theta_0^5} + 2\frac{w'''}{\theta_0^3} + \frac{w'}{\theta_0} = \left(\frac{MR}{EI}\right)\left(\frac{w'''}{\theta_0^3} + \frac{1}{\theta}w'\right) \quad (6)$$

$$\frac{w^V}{\theta_0^5} + 2\frac{w'''}{\theta_0^3} + \frac{w'}{\theta_0} = \left(\frac{qR^3}{EI}\right)\left(\frac{w'''}{\theta_0^3} + \frac{1}{\theta}w'\right) \quad (7)$$

The boundary conditions for both ends clamped, both ends simply supported and for mixed simply supported-clamped ends are, respectively.

$$w = w' = w'' = 0 \text{ at } X=0 \text{ and } 1 \quad (8)$$

$$w = w' = w''' = 0 \text{ at } X=0 \text{ and } 1 \quad (9)$$

$$\begin{aligned} w = w' = w''' = 0 \text{ at } X=0, \\ w = w' = w'' = 0 \text{ at } X=1 \end{aligned} \quad (10)$$

3. Differential Quadrature Method

The differential quadrature method (DQM) was introduced by Bellman and Casti [9]. By formulating the quadrature rule for a derivative as an analogous extension of quadrature for integrals in their introductory paper, they proposed the differential quadrature method as a new technique for the numerical solution of initial value problems of ordinary and partial differential equations. It was applied for the first time to static analysis of structural components by Jang *et al.* [11]. The versatility of the DQM to engineering analysis in general and to structural analysis in particular is becoming increasingly evident by the related publications of recent years. Kang and Han [12] applied the method to the analysis of a curved beam using classical and shear deformable beam theories, and Kang [13] studied the vibration analysis of curved beams using DQM. From a mathematical point of view, the application of the differential quadrature method to a partial differential equation can be expressed as follows:

$$\begin{aligned} L\{f(x)\}_i &= \sum_{j=1}^N W_{ij} f(x_j) \\ \text{for } i, j &= 1, 2, \dots, N \end{aligned} \quad (11)$$

where L denotes a differential operator, x_j are the discrete points considered in the domain, i are the row vectors of the N values, $f(x_j)$ are the function values at these points, W_{ij} are the weighting

coefficients attached to these function values, and N denotes the number of discrete points in the domain. This equation, thus, can be expressed as the derivatives of a function at a discrete point in terms of the function values at all discrete points in the variable domain.

The general form of the function $f(x)$ is taken as

$$f_k(X) = X^{k-1} \text{ for } k=1, 2, 3, \dots, N \quad (12)$$

If the differential operator L represents an n^{th} derivative, then

$$\sum_{j=1}^N W_{ij} x_j^{k-1} = (k-1)(k-2)\dots(k-n) x_i^{k-n-1} \quad (13)$$

for $i, k=1, 2, \dots, N$

This expression represents N sets of N linear algebraic equations, giving a unique solution for the weighting coefficients, W_{ij} , since the coefficient matrix is a Vandermonde matrix which always has an inverse.

4. Application

The DQM is applied to the determination of the in-plane buckling of the curved beams. The differential quadrature approximations of the governing equations and boundary conditions are shown.

Applying the differential quadrature method to equations (3), gives

$$\begin{aligned} \frac{1}{\theta_0^5} \sum_{j=1}^N E_{ij} w_j + \frac{2}{\theta_0^3} \sum_{j=1}^N C_{ij} w_j + \frac{1}{\theta_0} \sum_{j=1}^N A_{ij} w_j \\ = \left(\frac{qR^3}{EI} \right) \left(\frac{1}{\theta_0^3} \sum_{j=1}^N C_{ij} w_j \right) \end{aligned} \quad (14)$$

And applying the differential quadrature method to equations (5), gives

$$\begin{aligned} \frac{1}{\theta_0^5} \sum_{j=1}^N E_{ij} w_j + \frac{2}{\theta_0^3} \sum_{j=1}^N C_{ij} w_j + \frac{1}{\theta_0} \sum_{j=1}^N A_{ij} w_j \\ = \left(\frac{MR}{EI} \right) \left(\frac{1}{\theta_0^3} \sum_{j=1}^N C_{ij} w_j + \frac{2}{\theta_0} \sum_{j=1}^N A_{ij} w_j \right) \end{aligned} \quad (15)$$

where A_{ij} , C_{ij} , and E_{ij} are the weighting coefficients for the first-, third-, and fifth-order derivatives, respectively, along the dimensionless axis.

The boundary conditions for clamped ends, given by equations (8), can be expressed in differential quadrature form as follows:

$$\begin{aligned} w_1 &= 0 & \text{at } X=0 \\ w_N &= 0 & \text{at } X=1 \\ \sum_{j=1}^N A_{2j} w_j &= 0 & \text{at } X=0+\delta \\ \sum_{j=1}^N A_{(N-1)j} w_j &= 0 & \text{at } X=1-\delta \\ \sum_{j=1}^N B_{3j} w_j &= 0 & \text{at } X=0+2\delta \\ \sum_{j=1}^N B_{(N-2)j} w_j &= 0 & \text{at } X=1-2\delta \end{aligned} \quad (16)$$

Similarly, the boundary conditions for simply supported ends given by equations (9) can be expressed in differential quadrature form as follows:

$$\begin{aligned} w_1 &= 0 & \text{at } X=0 \\ w_N &= 0 & \text{at } X=1 \\ \sum_{j=1}^N A_{2j} w_j &= 0 & \text{at } X=0+\delta \\ \sum_{j=1}^N A_{(N-1)j} w_j &= 0 & \text{at } X=1-\delta \\ \sum_{j=1}^N C_{3j} w_j &= 0 & \text{at } X=0+2\delta \\ \sum_{j=1}^N C_{(N-2)j} w_j &= 0 & \text{at } X=1-2\delta \end{aligned} \quad (17)$$

where B_{ij} is the weighting coefficients for the second-order derivative. Here, δ denotes a very small distance measured along the dimensionless axis from the boundary ends. In their work on the application of DQM to the static analysis of beams and plates, Jang *et al.* [11] proposed the so-called δ -technique wherein adjacent to

the boundary points of the differential quadrature grid points chosen at a small distance. This δ approach is used to apply more than one boundary conditions at a given station.

The boundary conditions for one simply supported and one clamped ends, given by equations (10), can be expressed in differential quadrature form as

$$\begin{aligned} w_1 &= 0 & \text{at } X=0 \\ w_N &= 0 & \text{at } X=1 \\ \sum_{j=1}^N A_{2j} w_j &= 0 & \text{at } X=0+\delta \\ \sum_{j=1}^N A_{(N-1)j} w_j &= 0 & \text{at } X=1-\delta \\ \sum_{j=1}^N C_{(N-2)j} w_j &= 0 & \text{at } X=0+2\delta \\ \sum_{j=1}^N B_{3j} w_j &= 0 & \text{at } X=1-2\delta \end{aligned} \quad (18)$$

Mixed boundaries can be easily accommodated by combining these equations; simply change the weighting coefficients. While most analytical methods use the rather laborious technique of superposition to arrive at solutions for mixed boundary problems, this approach of breaking the problem into several easy is not required in the DQM. This set of equations together with the appropriate boundary conditions can be solved for the in-plane buckling.

5. Numerical results and comparisons

In-plane buckling parameters $M^* (= MR/EI)$ subjected to equal and opposite in-plane bending moments and $q^* (= qR^3/EI)$ subjected to uniformly distributed radial loads are calculated by the differential quadrature method (DQM) and are presented together with existing exact solutions. The values M^* and q^* are evaluated for the case of various end conditions and opening angles.

Table 1 presents the results of convergence studies

relative to the number of grid points N for the case of both ends simply supported with $\theta_0 = 180^\circ$. The data show that the accuracy of the numerical solution increases with increasing N . Then numerical instabilities arise if N becomes too large (possibly greater than approx. 17). Table 2 shows the sensitivity of the solution to the choice of δ for the case of both ends simply supported. The optimal value for δ is found to be 1×10^{-10} , which is obtained from trial-and-error calculations. The solution accuracy decreases due to numerical instabilities if δ becomes too big (possibly greater than approx. 1×10^{-5}).

[Table 1] Critical moment of in-plane buckling parameter, $M^* (= MR/EI)$, with both ends simply supported for a range of grid points N ; $\delta = 1 \times 10^{-10}$ and $\theta_0 = 180^\circ$

Exact (Kang and Yoo [7]) θ_0 (degree)	Number of grid points			
180°	9	11	13	15
-4.5000	-4.4900	-4.5002	-4.5000	-4.5006

[Table 2] Critical moment of in-plane buckling parameter, $M^* (= MR/EI)$, with both ends simply supported for a range of δ ; $N=13$ and $\theta_0 = 180^\circ$

Exact (Kang and Yoo [7]) θ_0 (degree)	δ			
$\theta_0 = 180^\circ$	1×10^{-6}	1×10^{-8}	1×10^{-10}	1×10^{-12}
-4.5000	-4.4584	-4.5002	-4.5000	-4.5003

In Table 3, the critical moments M^* determined by the DQM are compared with the exact solutions by Kang and Yoo [7] for the case of both ends simply supported. Table 4 shows the critical moments M^* determined by the DQM for the case of with both ends clamped and simply supported - clamped ends without comparison since no data are available.

[Table 3] Critical moment M^* ($= MR/EI$) of in-plane buckling of curved beams with both ends simply supported ; $N=13$ and $\delta = 1 \times 10^{-10}$

θ_0 (degree)	M^* ($= MR/EI$)	
	Exact (<i>Kang and Yoo</i> [7])	DQM
30	-144.01	-144.01
60	-36.029	-36.029
90	-16.071	-16.071
120	-9.1429	-9.1429
150	-6.0260	-6.0260
180	-4.5000	-4.5000

[Table 4] Critical moment M^* ($= MR/EI$) of in-plane buckling of curved beams with both ends clamped and simply supported - clamped ends ; $N=13$ and $\delta = 1 \times 10^{-10}$

θ_0 (degree)	DQM	
	Both ends clamped	Simply supported - clamped ends
30	-295.93	-234.96
60	-75.059	-59.175
90	-34.256	-26.656
120	-20.114	-15.318
150	-13.762	-10.094
180	-10.597	-7.1657

In Table 5, the critical loads q^* by the DQM are also compared with the exact solution by *Kang and Yoo* [7] for the case of both ends simply supported. Table 6 shows the critical loads q^* by the DQM for the case of both ends clamped and simply supported - clamped ends without comparison since no data are available.

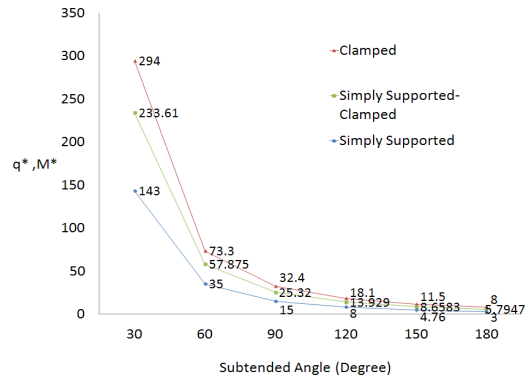
[Table 5] Critical load q^* ($= qR^3/EI$) of in-plane buckling of curved beams with both ends simply supported; $N=13$ and $\delta = 1 \times 10^{-10}$

θ_0 (degree)	q^* ($= qR^3/EI$)	
	Exact (<i>Kang and Yoo</i> [7])	DQM
30	-142.01	-142.17
60	-34.028	-34.021
90	-14.063	-14.063
120	-7.1111	-7.1122
150	-3.9336	-3.9336
180	-2.2500	-2.2500

[Table 6] Critical load q^* ($= qR^3/EI$) of in-plane buckling of curved beams with both ends clamped and simply supported - clamped ends ; $N=13$ and $\delta = 1 \times 10^{-10}$

θ_0 (degree)	DQM	
	Both ends clamped	Simply supported and clamped ends
30 ⁰	-292.71	-232.48
60 ⁰	-71.658	-56.615
90 ⁰	-30.782	-24.096
120 ⁰	-16.502	-12.742
150 ⁰	-9.9246	-7.5167
180 ⁰	-6.3862	-4.7129

Fig. 3 shows critical values of in-plane buckling parameters, q^* and M^* , using Eqs. (6) and (7) by DQM for the case of both ends simply supported, simply supported - clamped ends, and clamped.



[Fig. 3] Critical values of in-plane buckling parameters, q^* and M^* , using Eqs. (6) and (7) by DQM for the case of both ends simply supported, simply supported - clamped ends, and clamped.

From Tables 3-6, it is seen that the critical buckling parameters of the member with clamped ends are much higher than those of the member with simply supported ends and those of the member with mixed simply supported -clamped ends. The critical buckling parameters can be increased by decreasing the opening angle θ_0 . In general, the difference among the various studies are not substantial where the subtended angle is small. However, the differences become significant as the subtended angle becomes large. The DQM results show that the critical values q^* of buckling parameter by *Kang*

and Yoo [7] is lower than those by Yang and Kuo [5], and, but, the critical values M^* of buckling parameter by Kang and Yoo [7] is higher than those by Yang and Kuo [5]. As can be seen, the numerical results by the DQM show good to excellent agreement with the exact solutions.

6. Conclusions

The differential quadrature method (DQM) was applied to the computation of the eigenvalues of the equations governing in-plane buckling of the curved beams under equal and opposite bending moments and uniformly distributed radial loads. The present approach gives excellent results for the cases treated while requiring only a limited number of grid points: only thirteen discrete points were used for the evaluation. More results are given for two sets of boundary conditions not considered by previous investigators for in-plane buckling: clamped-clamped and simply supported- clamped ends.

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